Mathematical Methods in Physics HW1

1. Which of these form a group? If they do, identify which element acts as the identity. If they do not, specify which group criteria they do not meet.

- Integers with addition These do, and the identity is 0.
- Integers with multiplication These do not, since each element must have an inverse, and the multiplicative inverse for an integer is not an integer, e.g. $4 \times \frac{1}{4} = 1$. Also, 0 does not have a multiplicative inverse.
- Rationals with addition These do, and the identity is 0.
- Rationals with multiplication These do not, since there is no multiplicative inverse to 0.
- 3x3 matrices with arbitrary real elements with matrix multiplication These do not, since there are obviously 3x3 matrices without matrix multiplicative inverses, e.g. $M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ obviously has no inverse s.t. $.MM^{-1} = I$.
- 3x3 matrices with arbitrary real elements with addition These do, and the identity element is just $M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.
- Imaginary numbers with addition Well, this one depends on whether or not you imagine 0 as a real or imaginary number (and there are arguments and cases for each). If you imagine it being real, then no this does not form a group since the identity 0 would not be included. If you imagine it being imaginary, then yes since the identity is included.
- Imaginary numbers with multiplication Definitely not, since the identity 1 is not imaginary.

2. Which of these form a field? If they do then identify the field ingredients. If they do not, identify which ingredients go wrong.

- 2D rotations matrices with matrix addition and matrix multiplication These do not, since the elements with just addition do not form an abelian group. Adding two rotation matrices together does NOT give rise to another rotation matrix, e.g. $R_\pi + R_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (which is not a rotation matrix). So closure is not satisfied, and neither is the need for an additive identity $I_+ = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
- 2D diagonal matrices with real elements with matrix addition and matrix multiplication These do not. All of the elements with addition form an abelian group with identity $I_+ = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, but the elements without $I_+ = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ do not form an abelian group under multiplication since elements like $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ are allowed, but do not have an inverse.

- 2D arbitrary with real elements with matrix addition and matrix multiplication These will form an abelian group under addition, but will not form an abelian group under multiplication since in general 2x2 matrices don't commute, i.e. $AB \neq BA$.

3. Which of these constitute a vector space? If they do, show that they do. If they don't, show why they don't.

- An n-tuple of complex numbers over the field of real numbers These do since multiplication of a complex number (an element of the vector) by a real number (an element of the field) gives back a complex number, i.e. a component of the resulting vector.
- An n-tuple of imaginary numbers over the field of real numbers This one depends on whether you cast 0 as real or imaginary (both are possible). If real, then no since the identity in the additive group is missing. If imaginary, then yes since the additive group includes the identity, and multiplying an imaginary number (an element of the vector) by a real number (an element of the field) results in an imaginary number, i.e. a component of the resulting vector.
- An n-tuple of imaginary numbers over the field of complex numbers This will not since multiplying an imaginary number (an element of the vector) by a complex number (an element of the field) results in a complex number, i.e. a component of the resulting vector which is now complex and not a part of the original set.
- An n-tuple of complex numbers over the field of imaginary numbers Actually this stops at the question since imaginary numbers do not form a field! Even if we define 0 to be imaginary and serve as the identity to form an abelian group under addition, we cannot include 1 as the identity to form an abelian group under multiplication since 1 is definitely not real.

4. Show that even and odd integers do not form a group under multiplication. So the multiplication (or Cayley) table is

$$\begin{array}{c|cc} \times & E & O \\ \hline E & E & E \\ O & E & O \end{array}$$

which implies that the identity is E since only for the identity does $E \times E = E$. But this disagrees with $E \times O = E$, since E is clearly not acting as the identity.

5. Consider $\mathbb{R}^3$ and a usual set of orthonormal basis vectors, $\hat{\imath}, \hat{\jmath}, \hat{k}$. Show that the set $\hat{a} = \hat{\imath}, \hat{b} = \hat{\jmath}, \hat{c} = \frac{1}{3}(\hat{\imath} + \hat{\jmath} + \hat{k})$ forms a basis, though not an orthonormal one (well at least it is normalized though not orthogonal). Let's consider $x = \alpha_1 \hat{a} + \alpha_2 \hat{b} + \alpha_3 \hat{c}$. To demonstrate linear independence, we need to show that $x = 0 \Rightarrow \alpha_i = 0$. Consider x in the original basis: $x = \alpha_1 \hat{\imath} + \alpha_2 \hat{\jmath} + \alpha_3 \frac{1}{3}(\hat{\imath} + \hat{\jmath} + \hat{k}) = (\alpha_1 + \frac{1}{3}\alpha_3)\hat{\imath} + (\alpha_2 + \frac{1}{3}\alpha_3)\hat{\jmath} + \frac{1}{3}\alpha_3\hat{k}$. Now we know that $\hat{\imath}, \hat{\jmath}, \hat{k}$ is a basis, and so these are linearly independent. This means that for any $x = \beta_1 \hat{\imath} + \beta_2 \hat{\jmath} + \beta_3 \hat{k} = 0 \Rightarrow \beta_i = 0$. Therefore: $\beta_1 = \alpha_1 + \frac{1}{3}\alpha_3 = 0, \beta_2 = \alpha_2 + \frac{1}{3}\alpha_3 = 0$ and $\beta_3 = \frac{1}{3}\alpha_3 = 0$. The last one tells us that $\beta_3 = 0$ and hence $\beta_1 = \beta_2 = 0$ from the first two. So this set is linearly independent. To argue that it forms a basis, we also need to show that it spans the space. Well, the space is 3-dimensional and there are 3 of them so yes it does.

6. Show that all groups with only three elements are isomorphic. How many different (non-isomorphic) variations of four element groups are there? Consider the multiplication table

$$\begin{array}{c|ccc} * & I & A & B \\ \hline I & I & A & B \\ A & A & & \\ B & B & & \end{array}$$

. Clearly the only thing left to be determined is the lower 2x2 contents since the rest is just the identity acting on the two nontrivial elements.

Let's consider what can go on the diagonal.

If $A * A = A$, then $A = I$ which is not good.

If $A * A = I$, then $A = A^{-1}$, but what is $A * B =$? It can't be the identity since that would imply that $A^{-1} = B$. It can't be A since that would imply that $B = I$, and it can't be B since that would imply that $A = I$. So no go on this choice of diagonals.

Finally, if $A * A = B$, then A^{-1} must be B (since there is nothing else left) hence $A * B = I$. However then $A * A = B \Rightarrow (A * A) * B = A * (A * B) = A = B * B$. This works out and is the only choice. So

$$\begin{array}{c|ccc} * & I & A & B \\ \hline I & I & A & B \\ A & A & B & I \\ B & B & I & A \end{array}.$$

For four component groups we can consider $\{I, A, B, C\}$ and begin by asking, what are the inverses to A, B and C. It turns out there are two possibilities:

1) $A^{-1} = A, B^{-1} = B, C^{-1} = C$ which then leads to the question what is $A * B, B * A, A * C, C * A, B * C, C * B$? Well if $A * B = A$ or B, then that means that B or A are acting as the identity. So that only leaves $A * B = C = B * A$, and similarly $A * C = B = C * A$ and $B * C = A = C * B$. Leaving us with:

	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

2) $A^{-1} = B$ hence $B^{-1} = A$ and $C^{-1} = C$. Obviously we could pick any two as inverses to each other. Considering the rest of the combinations, obviously $A * B = I = B * A$, and then that leaves $A * C = B = C * A$ and $B * C = A = C * B$ by arguments similar to those above. And finally $A * A = C$ and $B * B = C$ works out with the other conditions, whereas the alternative $A * A = B$ and $B * B = A$ does not. To see this take the first one and apply B to the left and (the equivalent) $C * A$ to the right. This gives $B * A * A = C * A * B$, but using that $B * A = A * B = I$, this becomes $A = C$. Leaving us with:

	I	A	B	C
I	I	A	B	C
A	A	C	I	B
B	B	I	C	A
C	C	B	A	I

7. Prove that if the set of vectors $\{x_i\}$ is a basis for a vector space V, then for c_i arbitrary nonzero scalars, so is the set $\{c_ix_i\}$. Clearly if $\{x_i\}$ span the space then so does $\{c_ix_i\}$. The trick is showing that the set $\{c_ix_i\}$. is linearly independent. Well recall that if $\{x_i\}$ is linearly independent then $\sum_i \alpha_i x_i = 0 \Rightarrow \alpha_i = 0$. Cleverly inserting the number 1 into the summation we have if $\sum_i \frac{\alpha_i}{c_i} c_i x_i = 0 \Rightarrow \alpha_i = 0$, which is obviously still true. But if $\alpha_i = 0$ then so does $\frac{\alpha_i}{c_i} = 0$ for $c_i \neq 0$. So in the end we have $\sum_i \frac{\alpha_i}{c_i} c_i x_i = 0 \Rightarrow \frac{\alpha_i}{c_i} = 0$ which tells us that the set $\{c_ix_i\}$ is linearly independent. So it forms a basis as well.